ON SHOCK WAVES IN THE FLOW OF A POLYTROPIC GAS, HAVING STRAIGHT - LINE CHARACTERISTICS

(OB UDARNYKH VOLNAKH V TECHENIIAKH POLITROPNOGO GAZA, IMEIUSHCHIKH PRIAMOLINEINIE KHARAKTERISTIKI)

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A. F. SIDOROV
(Cheliabinsk)

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1. We shall consider the two-dimensional unsteady motion of a polytropic gas with the equation of state $p = a^2(S)\rho$, where p is the pressure, ρ the density, S the entropy and γ the adiabatic exponent.

Let $c = \sqrt{(dp/d\rho)}$ be the sound speed, and u_1, u_2 the components of the velocity vector $\mathbf{u}$.

In [1] was studied isentropic flow having straight-line characteristics in the $x_1 x_2 t$-space

$$\frac{dx_1}{\Delta_1} = \frac{dx_2}{\Delta_2} = \frac{dt}{1} \qquad (1.1)$$

The quantities Δ_1 and Δ_2, and also the functions u_1 and u_2, are assumed to depend on two parameters α_1 and α_2, which determine the position of the characteristics.

In what follows we shall study mainly flows which are not simple waves or conical flows; by virtue of the theorems established in [1], such flows will be potential flows for $\gamma \neq 2$ (for $\gamma = 2$, rotational flows may also have straight characteristics). In this paper we shall give a method of finding the flow of a gas behind shock waves of constant strength moving into a uniform region, for a class of sufficiently smooth (in some sense) shock fronts, when the flow behind the shock fronts has straight characteristics.

For simplicity we shall consider an isothermal gas, though all the arguments may be carried over to the adiabatic case with arbitrary γ, for which we merely state a few results.

In the adiabatic case, the constancy of the normal velocity of the shock front follows at once from the Hugoniot conditions as soon as we are given a constant region ahead of the shock, and we assume isentropic flow behind it. In the isothermal gas, this same property is also deduced from the Hugoniot conditions, in addition to the assumption that the flow behind the shock wave possesses straight characteristics.

2. In the equation of state of an isothermal gas $p = a^2\rho$, where $a^2 = RT$, we assume for simplicity $a^2 = 1$ (a^2 is the square of the sound speed, which is a constant in an isothermal gas).

The flow of an isothermal gas with straight characteristics is described by the system of equations [1]

$$\Delta_i = u_i + q_i \qquad (i = 1, 2) \tag{2.1}$$

$$(1 - q_1^2)(q_{22} + 1) + 2q_1q_2q_{12} + (1 - q_2^2)(q_{11} + 1) = 0 \tag{2.2}$$

$$(1 - q_1^2)\,\partial u_1 / \partial\alpha_1 - 2q_1q_2\,\partial u_1 / \partial\alpha_2 + (1 - q_2^2)\,\partial u_2 / \partial\alpha_2 = 0 \tag{2.3}$$

$$\partial u_1 / \partial\alpha_2 = \partial u_2 / \partial\alpha_1 \tag{2.4}$$

$$x_i - \Delta_i(\alpha_1, \alpha_2)\, t = \alpha_i \qquad (i = 1, 2) \tag{2.5}$$

Here

$$q = \ln\rho, \qquad q_i = \frac{\partial q}{\partial u_i}, \qquad q_{ik} = \frac{\partial^2 q}{\partial u_i \partial u_k} \tag{2.6}$$

Equation (2.5) serves to determine the functions u_i and q in the x_1x_2t-space.

We consider a stationary gas with $u_1 = u_2 = 0$ and $\rho = 1$, into which the shock wave enters. The equation of the shock surface may be taken in the form

$$\alpha_2 - f(\alpha_1) = 0 \tag{2.7}$$

The equation of the shock surface $\Phi(a_1, a_2) = 0$ may be considered as general, insofar as a given equation of a surface $F(x_1, x_2, t) = 0$ reduces to it, after using supplementary conditions between u_1, u_2 and q as functions of a_1 and a_2 resulting from Hugoniot's conditions. The Hugoniot conditions in the case considered are written as:

$$e^q = D^2, \qquad |\mathbf{u}| = D - \frac{1}{D}, \qquad \mathbf{u}\cdot\mathbf{t} = 0 \tag{2.8}$$

where D is the speed of the shock front and $\mathbf{t}$ a tangential vector to the shock front.

The method of studying conditions (2.8) is analogous to the method applied in [1] for deriving the basic equations. With the help of Formulas (2.1) and (2.5), Equation (2.7) and the formula

$$D = \frac{\partial\alpha_2 / \partial t - f'\,\partial\alpha_1 / \partial t}{\sqrt{(\partial\alpha_2 / \partial x_1 - f'\,\partial\alpha_1 / \partial x_1)^2 + (\partial\alpha_2 / \partial x_2 - f'\,\partial\alpha_1 / \partial x_2)^2}} \tag{2.9}$$

condition (2.8) may be reduced to the form

$$A_i + tB_i = 0 \qquad (i = 1, 3)$$

where A_i and B_i are functions of α_1 alone, and consequently, since t is arbitrary, each must be zero. Thus, we obtain four equations:

$$u_1 + u_2 f' = 0 \tag{2.10}$$

$$(p_{11} + f'p_{12}) u_1 + (p_{21} + f'p_{22}) u_2 = 0 \qquad \left(p_{ij} = \frac{\partial \Delta_i}{\partial \alpha_j}\right) \tag{2.11}$$

$$f' e^{q/2} + \frac{u_1(-\Delta_2 + f'\Delta_1)}{\sqrt{u_1^2 + u_2^2}} = 0 \tag{2.12}$$

$$(p_{21} + f'p_{22}) e^{q/2} + \frac{u_1 [\Delta_1 (p_{21} + f'p_{22}) - \Delta_2 (p_{11} + f'p_{12})]}{\sqrt{u^2 + u_2^2}} = 0 \tag{2.13}$$

Finally, elimination of D from (2.8) gives yet another result:

$$e^{q/2} = \frac{1}{2}\left(\sqrt{u_1^2 + u_2^2} + \sqrt{u_1^2 + u_2^2 + 4}\right) \tag{2.14}$$

We shall prove that $|\mathbf{u}| = \sqrt{F} =$ const along the shock front. To this end we calculate along the front the following:

$$q = \int q_1 \, du_1 + q_2 \, du_2$$

Using (2.1) and (2.12), and also the fact that along the front

$$du_i = \left(\frac{\partial u_i}{\partial \alpha_1} + \frac{\partial u_i}{\partial a_2} f'\right) d\alpha_1$$

we reduce the expression for q to the form

$$q = -u_1^2 - u_2^2 - \frac{1}{2}\sqrt{(u_1^2 + u_2^2)(u_1^2 + u_2^2 + 4)} + Q \qquad (Q = \text{const}) \tag{2.15}$$

Comparing (2.15) with (2.14), we see that $u_1^2 + u_2^2 = F =$ const along the front, the constant Q being determined by the given $|\mathbf{u}|$ on the front.

Thus, in the class of flows of an isothermal gas which we consider, shock waves can propagate only with constant speed D, and consequently the quantities $|\mathbf{u}|$ and q are constant along the front.

Moreover, analysis of Equations (2.10) to (2.13) shows that the following relations obtain along the front:

$$u_1|_{\alpha_2 = f(\alpha_1)} = -f'\sqrt{\frac{F}{1 + f'^2}}, \qquad u_2|_{\alpha_2 = f(\alpha_1)} = \sqrt{\frac{F}{1 + f'^2}} \tag{2.16}$$

$$q_1|_{u_1^2 + u_2^2 = F} = \frac{G}{F} u_1, \qquad q_2|_{u_1^2 + u_2^2 = F} = \frac{G}{F} u_2 \tag{2.17}$$

Here

$$F = \left(D - \frac{1}{D}\right)^2, \qquad G = G_1 = -\frac{1}{2}\left(3F + \sqrt{F(F+4)}\right) \quad \text{or}$$

$$G = G_2 = \frac{1}{2}\left(\sqrt{F(F+4)} - F\right) \tag{2.18}$$

Formulas (2.16) provide the initial data on the shock front for Equations (2.3) and (2.4) in the a_1-a_2-plane, and Formulas (2.17) give the initial data for Equation (2.2) in the plane of the velocity components. Equation (2.2) and the system (2.3), (2.4) for the functions u_1 and u_2 are hyperbolic in the neighborhood of the line $u_1^2 + u_2^2 = F$ when $G = G_1$, and, generally speaking, are elliptic there when $G = G_2$. The choice of the sign in the formulas for u_1 and u_2 is fixed by the direction of propagation of the shock wave. The shape of the front at the initial instant, given by the function $f(a_1)$, may be arbitrary. We observe that for conical flow ($\Delta_1 = a_1$, $\Delta_2 = a_2$) the shape of the front will not be arbitrary, but must be either plane or cylindrical. This follows from Equations (2.10) to (2.13).

The function q, which satisfies condition (2.17), is uniquely determined from the equation

$$2q''u + 2q' - 2q'^2u - 4q'^3u + 1 = 0 \qquad (q = q(u), \quad u = u_1^2 + u_2^2) \tag{2.19}$$

and from condition (2.17) for Equation (2.9), the initial conditions for the Cauchy problem being

$$q'|_{u=F} = \frac{G}{2F}, \qquad q|_{u=F} = 2\ln\frac{\sqrt{F} + \sqrt{F+4}}{2} \tag{2.20}$$

We remark further that the function q does not depend on the shape of the shock wave, but is determined only by the speed of its propagation into a constant region.

By means of a velocity potential Φ, Equations (2.3) and (2.4) may be reduced to a single equation of the second order.

As an example of that, we use Legendre's transformation and introduce polar coordinates $u_1 = r\cos\phi$, $u_2 = r\sin\phi$; then we obtain a linear second-order equation for Φ^o of the form

$$\frac{\partial^2\Phi^\circ}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Phi^\circ}{\partial\varphi^2} + \frac{1}{r}\frac{\partial\Phi^\circ}{\partial r} - 4q'^2(r^2)\left(\frac{\partial^2\Phi^\circ}{\partial\varphi^2} + r\frac{\partial\Phi^\circ}{\partial r}\right) = 0 \tag{2.21}$$

for which

$$\Phi^\circ = \alpha_1 u_1 + \alpha_2 u_2 - \Phi, \qquad \frac{\partial\Phi^\circ}{\partial u_1} = \alpha_1, \qquad \frac{\partial\Phi^\circ}{\partial u_2} = \alpha_2$$

From condition (2.16) for Equation (2.21) we set up the following problem:

$$\frac{\partial \Phi^\circ}{\partial r}\Big|_{r=\sqrt{F}} = \cos\varphi f'^{-1}(-\cot\varphi) + \sin\varphi f(f'^{-1}(-\cot\varphi)) = l(\varphi)$$
$$\sqrt{F}\frac{\partial \Phi^\circ}{\partial r} - \Phi^\circ|_{r=\sqrt{F}} = 0 \tag{2.22}$$

Here f'^{-1} represents the inverse function to f'. In the coefficients of Equation (2.21) the variable ϕ does not enter explicitly; thus we may apply the method of Fourier for the solution. Seeking a solution in the form $\Phi^\circ = \psi(\phi)\chi(r)$, we obtain equations for ψ and χ:

$$\psi'' - \lambda\psi = 0 \qquad \chi'' - \left(4rq'^2 - \frac{1}{r}\right)\chi' - \lambda\left(4q'^2 - \frac{1}{r^2}\right)\chi = 0 \tag{2.23}$$

where λ is a constant. Thus, Φ° may be taken in the form

$$\Phi^\circ = \sum_\lambda a_\lambda \psi_\lambda(\varphi)\, x_\lambda(r)$$

where a_λ are arbitrary constants.

For some concrete gasdynamical-flow problems (when $\lambda = -k^2$, k integer), with the assumptions of the boundedness of $\chi_\lambda(\sqrt{F})$ for all λ and sufficient smoothness of the function $l(\phi)$, it is possible to justify Fourier's method for Equation (2.21) in the hyperbolic case and to prove the convergence of the corresponding series.

3. As an example, we solve the problem for an isothermal gas assuming that, at the initial instant, the shock is elliptical in shape

$$x_1^2/a^2 + x_2^2/b^2 = 1 \tag{3.1}$$

and the gas in front of the shock (i.e. inside the ellipse) is at rest, with $\rho = 1$.

For $t = 0$ we have $a_i = x_i$, and it is sufficient to find the distribution of the velocities u_1 and u_2 behind the shock wave at the instant $t = 0$. After that, the flow in the $x_1 - x_2 - t$ space is found from Formula (2.5). We observe that in our consideration we assume the hyperbolicity of Equation (2.21) behind the shock.

The lines $x_1 = 0$ and $x_2 = 0$, by symmetry, may be considered as rigid walls. Boundary conditions on the walls $x_1 = 0$ and $x_2 = 0$ for the potential Φ° are set as:

$$\frac{\partial \Phi^\circ}{\partial u_1}\Big|_{u_1=0} = 0, \qquad \frac{\partial \Phi^\circ}{\partial u_2}\Big|_{u_2=0} = 0 \tag{3.2}$$

In this case

$$l(\varphi) = \sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi} \tag{3.3}$$

Solving this problem by Fourier's method, we obtain

$$\lambda = -(2m)^2, \qquad m = 0, 1, 2, \ldots$$

and for the velocity potential Φ° we obtain the expression

$$\Phi^\circ = s_0 \chi_0(r) + \sum_{m=0}^{\infty} s_{2m} \chi_{2m}(r) \cos 2m\varphi \tag{3.4}$$

where

$$s_0 = \frac{1}{\pi} \int_0^\pi \sqrt{a^2 \cos^2 \frac{t}{2} + b^2 \sin \frac{t}{2}} dt \quad s_{2m} = \frac{2}{\pi} \int_0^\pi \sqrt{a^2 \cos^2 \frac{t}{2} + b^2 \sin^2 \frac{t}{2}} \cos mt \, dt \tag{3.5}$$

i.e. coefficients are expressed as complete elliptic integrals, and the initial conditions for $\chi_{2m}(r)$ follow:

$$\chi_{2m}\big|_{r=\sqrt{F}} = \sqrt{F}, \qquad \chi'_{2m}\big|_{r=\sqrt{F}} = 1 \tag{3.6}$$

Functions χ_{2m} are found by numerical integration of the second equation (2.23) after determining q by numerical integration of Equation (2.19).

The dependence of functions u_1 and u_2 on x_1 and x_2 $(t = 0)$ may be found from the relations

$$\partial \Phi^\circ / \partial u_1 = x_1, \qquad \partial \Phi^\circ / \partial u_2 = x_2 \tag{3.7}$$

We note that the motion of the shock wave toward the center of the ellipse can only be considered up to the instant $t = t_0$, at which time the normals to the shock front begin to intersect and the shock front becomes broken.

4. We consider, in conclusion, flows with straight characteristics and shock waves in the adiabatic case.

These flows are described by the equations [1]

$$\Delta_i = u_i + c\theta_i \qquad \left(\theta = \frac{2}{\gamma - 1} c, c^2 = \left[\frac{\partial p}{\partial \rho}\right]_s\right) \tag{4.1}$$

$$\frac{\gamma - 1}{2} \theta \left[(1 - \theta_1^2) \theta_{22} + 2\theta_1\theta_2\theta_{12} + (1 - \theta_2^2) \theta_{11}\right] + \frac{\gamma - 3}{2} (\theta_1^2 + \theta_2^2) + 2 = 0 \tag{4.2}$$

$$(1 - \theta_1^2) \partial u_1 / \partial \alpha_1 - 2\theta_1\theta_2 \partial u_1 / \partial \alpha_2 + (1 - \theta_2^2) \partial u_2 / \partial \alpha_2 = 0 \tag{4.3}$$

$$\partial u_1 / \partial \alpha_2 = \partial u_2 / \partial \alpha_1 \tag{4.4}$$

$$x_i - \Delta_i(\alpha_1, \alpha_2)\, t = \alpha_i \qquad (i = 1,2) \tag{4.5}$$

Here

$$\theta_i = \frac{\partial \theta}{\partial u_i}, \qquad \theta_{ik} = \frac{\partial^2 \theta}{\partial u_i \partial u_k}$$

As before, we consider shock waves entering a stationary gas and shock fronts given by (2.7). The Hugoniot conditions in this case appear thus:

$$\mathbf{u} \cdot \mathbf{t} = 0 \tag{4.6}$$

$$\rho_1 (u_{1n} - D) = -\rho_0 D \tag{4.7}$$

$$p_1 + \rho_1 (u_{1n} - D)^2 = p_0 + \rho_0 D^2 \tag{4.8}$$

$$w_1 + \frac{1}{2}(u_{1n} - D)^2 = w_0 + \frac{1}{2} D^2, \qquad u_{1n} = \sqrt{u_1^2 + u_2^2} \tag{4.9}$$

Here D is the shock speed, quantities with index 1 or 0 refer respectively to the states behind or in front of the shock, w is the enthalpy.

Thus, if given the equation of state and ρ_0, Equations (4.7), (4.8) and (4.9) represent a system for the determination of the quantities u_{1n}, ρ and D, which, consequently, will appear constant in this case. The difference from the isothermal gas lies in the fact that after giving the state before the shock front the shock speed will be uniquely determined as soon as the entropy constant $a^2(S)$ is known. With the help of the expression

$$D = -\frac{u_1 \Delta + u_2 \Delta_2}{\sqrt{u_1^2 + u_2^2}} = \mathrm{const} \tag{4.10}$$

and condition (4.6), we obtain, in a manner analogous to the case of the isothermal gas, the following relations which are satisfied along the shock front:

$$u_1 |_{\alpha_2 = f(\alpha_1)} = -f' \sqrt{\frac{F}{1 + f'^2}} \qquad u_2 |_{\alpha_2 = f(\alpha_1)} = \sqrt{\frac{F}{1 + f'^2}} \tag{4.11}$$

$$\theta_1 |_{u_1^2 + u_2^2 = F} = \frac{C}{F} u_1, \qquad \theta_2 |_{u_1^2 + u_2^2 = F} = \frac{C}{F} u_2 \tag{4.12}$$

in which $C = u_1\theta_1 + u_2\theta_2 = \mathrm{const}$ appears from condition (4.10). Seeking a function θ in the form $\theta = \theta(u_1^2 + u_2^2)$ and letting $u = u_1^2 + u_2^2$, we obtain the second-order equation for the function θ

$$(\gamma - 1)\, \theta\, (\theta' + \theta'' u - 2\theta'^3 u) + (\gamma - 3)\, \theta'^2 u + 1 = 0 \tag{4.13}$$

Here

$$\theta' |_{u = F} = \frac{C}{2F}$$

and $\theta(F)$ is determined from Hugoniot conditions. Just as in the case of the isothermal gas, the function $\theta(u)$ is invariant with respect to the shape of the shock front. This problem with the initial data (4.11) for the potential $\Phi(\partial\Phi/\partial a = u_i)$ can be solved by Fourier's method in a completely analogous manner to the isothermal case.

Thus, by means of the method indicated it is possible, in the adiabatic case also, to obtain exact solutions to some gasdynamical problems with shock waves.

Furthermore, the method considered gives the possibility in some cases, both for isothermal and polytropic gases, of solving the problem of the motion of a curved piston which drives a shock wave in front of it, under the assumptions of sufficient smoothness (in some sense) of the piston shape at the initial instant. Thus, we can obtain some generalization to the curved piston of the solution of L.I. Sedov for a constant-speed cylindrical piston. These questions will be considered in subsequent papers. The exact solutions obtained, moreover, may be used as criteria of accuracy for numerical methods.

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